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On linking of cycles in locally connected spaces

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Abstract

The paper is devoted to studying the linking of cycles with compacta in LC^n-spaces and in particular homology Z-sets. The main two consequences of our considerations are the following:

(1) It is proved that a k-dimensional polyhedron cannot link a $(n-k-1)$-dimensional cycle in an n-dimensional Menger manifold.

(2) It is proved that a compact set in an ENR is a homology Z-set provided all its points are homology Z-sets. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

A cycle z of a space M lying in the complement of a compact set $X \subset M$ is said to be *linked* with X in M if z is homologically trivial in M and nontrivial in $M \setminus X$.

In Euclidean spaces (more generally in homology manifolds [3]) the dimension of a compactum is determined by the maximum of the dimensions of cycles which link with this compactum in some open set. Namely, a k-dimensional compactum in an n-dimensional space is locally linked with some cycle of *complementary* dimension $(n-k-1)$ and it is unlinked with any cycle of smaller dimension. This statement is known as the Alexandroff's Theorem on Obstructions in homology dimension theory [1].

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For general spaces one has the Second Fundamental Alexandroff's Theorem [1] that characterizes the dimension of compact spaces in terms of the separation of homologies. Its equivalent form in terms of the linking of cycles can be formulated as follows.

Theorem 1.1 [1]. *If a compactum X has dimension n then*

(1) *Every essential Vietoris cycle is linked with some compactum of complementary dimension.*
(2) *For every $k < n$ there exists a Vietoris k-cycle which is not linked with any compactum of the dimension less than $n - k - 1$.*

In this paper we are interested in the linking of cycles with compacta in Menger manifolds. Our first result in a natural way strengthens the second statement of the above theorem for Menger manifolds.

Theorem 1.2. *Let M be an n-dimensional Menger manifold, G be an Abelian group and C be a compactum in M of dimension $k < n$. Then C is not linked in M with any cycle over G of dimension less than $n - k - 1$.*

According to the above mentioned theorems finding cycles of complementary dimension linked with a given subset is of special interest.

Definition. A subset X of a space M is said to be *colinked* in M if there exists an Abelian group G and a cycle over G of dimension less than or equal to $\dim M - \dim X - 1$, which is linked with X in M. Otherwise the set X is said to be *uncolinked* in M.

For example, every subset of the boundary of a manifold is uncolinked in the manifold. More generally every Z-set of a space is uncolinked in it. Since every finite-dimensional compactum is embeddable in a Menger manifold as a Z-set it has uncolinked embeddings in Menger manifolds. Each singleton of any Menger manifold is a Z-set [2]. So a singleton does not admit any colinked embedding into any Menger manifold. On the other hand Theorem 1.1 implies the existence of compacta of any dimension admitting colinked embeddings in Menger manifolds.

The problem we are interested in is to recognize compacta which admit colinked embeddings into some Menger manifold. As the following theorem shows, this class of compacta does not include polyhedra.

Theorem 1.3. *Any polyhedron in any Menger manifold is uncolinked.*

This theorem motivates the following definition.

Definition. A space M is said to be *μ-uncolinked* if, for every embedding of M into any Menger manifold, its image is uncolinked, otherwise it is said to be μ-colinked.

Thus, all polyhedra are μ-uncolinked. The following theorem describes μ-uncolinkedness of 0-dimensional compacta.

Theorem 1.4. *A* 0*-dimensional compactum is* μ*-uncolinked iff it is countable.*

In Section 3 we prove that Pontrjagin surfaces and the Sierpinski gasket are μ-uncolinked, while the Sierpinski carpet is μ-colinked. We have succeeded the proof of μ-uncolinkedness of ENRs (ENR = finite-dimensional ANR) only in dimension 1. We are sure, however, that this is true in all dimensions.

Conjecture. Every ENR-compactum is μ-uncolinked.

Our results on Menger manifolds are based on the results of the Section 2, where the general theory of homology Z-sets in LC^n-spaces is developed. One can define a *homology* Z*-set* as a set which is not linked with any cycle in any open set. This theory is of an independent interest. The concept of a homology Z-set is presented in the formulation of Edwards' Free-Set Z-Set conjecture, which implies the Hilbert–Smith conjecture [6].

FSZS Conjecture. *Given any action by a cantor group on an ENR, the free set of the action is a homology* Z*-set.*

In Section 2, in particular, we prove that a compactum in an ENR-space is a homology Z-set iff all its points are homology Z-sets. So it is possible to replace "Set" by "Point" in the formulation of the Free-Set Z-Set conjecture. Edwards has informed us that he also has proved this reduction.

To give an exact formulation of our result we need the following definition.

Definition. A nowhere dense subset of a space which is not linked with any cycle of dimension $\leqslant k$ with the coefficient group G in any open set is said to be a Z_k^G*-set*.

The main result of Section 2 is the following theorem.

Theorem 1.5. *If all points of an* n*-dimensional compactum* C *are* Z_k^G*-sets in some* LC^{k+1}*-space* M *then* C *is a* Z_{k-n}^G*-set in* M.

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2. Homology Z-sets

In this section G denotes an Abelian group. All homologies are singular and have G as the coefficient group. All spaces are metrizable. We will be concerned with the cycles of dimension $\leqslant n$ of LC^n-spaces. In this case the singular homology groups are naturally isomorphic to the Čech homology groups up to dimension n (see [4]) and thus are continuous as well as exact.

Lemma 2.1. *Let C be a closed nowhere dense subset of a metric LC^{k+1}-space M. Then the following conditions are equivalent*:

(1) *C is a Z_k^G-set.*

(2) *For every open U and every $i \leqslant k+1$ one has $H_i(U, U \setminus C; G) = 0$.*

Proof. (2) ⇒ (1) Let us consider the exact sequence of the pair $(U, U \setminus C)$, where U is some open set. The condition $H_i(U, U \setminus C; G) = 0$ implies that the homomorphism $H_i(U \setminus C; G) \to H_i(U; G)$ is a monomorphism for each $i \leqslant k$. But this means that C is unlinked with any cycle of dimension less or equal k in U.

The implication (1) ⇒ (2) is proved by the classical Alexandroff construction of ε-modification of complexes [1]. Let us sketch this construction. Let z be a singular relative cycle of dimension $i \leqslant k+1$ of the pair $(U, U \setminus C)$. Then there is a polyhedral pair (P, P') of dimension i and a map $f : (P, P') \to (U, U \setminus C)$ such that $z = f_*(z')$ for some $z' \in H_i(P, P'; G)$. Since C is nowhere dense, without loss of generality we can suppose that the image of the 0-dimensional skeleton $f(P^{(0)})$ does not intersect C. For every r-dimensional simplex t_j^r of P one constructs a singular chain $\tilde{t}_j^r$ (its modification) in $U \setminus C$ of the same dimension in such a way that the equality $\partial t_i^r = \sum c_i^j t_j^{r-1}$ implies the equality $\partial \tilde{t}_i^r = \sum c_i^j \tilde{t}_j^{r-1}$ for all simplices.

The triangulation of P is supposed to be sufficiently fine to provide that the images of the modifications of all chains $\tilde{t}_j^r$ in the process of the above construction have diameters small enough to guarantee their homotopical triviality in U (here the condition LC^{k+1} is applied).

The modification of the 0-skeleton is defined by equalities $\tilde{t}_i^0 = t_i^0$. Let the modification be already defined for all simplices of dimension $< i$. Consider a simplex t_j^i of dimension i. The modification of its boundary has a sufficiently small diameter and therefore is homotopically and homologically trivial in some small open set U. Since C is a Z_k^G-set, it is not linked with the modification in U. Let $\tilde{t}_j^i$ be a cycle in $U \setminus C$ which bounds the modification of the boundary of t_j^i. The same is done with all simplices of dimension i. This completes the induction step. It is not difficult to see, and it was proved by Alexandroff [1] that an ε-modification of a cycle is homologous to the cycle. So each relative cycle of dimension $\leqslant k+1$ in U is homologous to a cycle (= its modification) in $U \setminus C$ and, hence, is trivial in $H_i(U, U \setminus C; G)$. □

Remark. The implication (2) ⇒ (1) of Lemma 2.1 holds without the assumption of the local $(k+1)$-connectedness.

Lemma 2.2. *A point x in an LC^{k+1}-space M is a Z_k^G-set iff $H_i(M, M \setminus x; G) = 0$ for all $i \leqslant k+1$.*

Proof. For every open U by the excision axiom one has $H_i(M, M \setminus x; G) \approx H_i(U, U \setminus x; G)$. Therefore by virtue of Lemma 2.1 the triviality of the homology $H_i(M, M \setminus x; G)$ is equivalent to the fact that x is a Z_k^G-set. □

Lemma 2.3. *Let C be a closed subset of an LC^k-space X and U be an open neighbourhood of C. Then C is a Z_k^G-set in X iff C is a Z_k^G-set in U.*

Proof. If C is a Z_k^G-set in X it is obvious that C is a Z_k^G-set in each of its neighborhoods. Suppose that C is a Z_k^G-set in U. First let us prove that C is not linked in X with any cycle z of dimension $\leqslant k$. Let $z = \partial z'$ be a cycle in $X \setminus C$ where z' is a chain in X. If z is not homologous to 0 in $X \setminus C$, then z' is a nontrivial relative cycle of the pair $(X, X \setminus C)$. By the excision axiom the natural homomorphism

$$e : H_*(U, U \setminus C; G) \to H_*(X, X \setminus C; G)$$

is an isomorphism. Let us consider $z'' = e^{-1}z'$. In this case $\partial z''$ is a cycle in $U \setminus C$, which is homologous to z in $X \setminus C$. But $\partial z''$ is homologous to 0 in $U \setminus C$, because C is a Z_k^G-set in U. Therefore z is homologous to 0 in $X \setminus C$, i.e., is not linked with C in X.

Consider now an open $X' \subset X$. The intersection $C' = X' \cap C$ is closed in X' and is easily seen to be a Z_k^G-set in $U' = U \cap X'$. Now the same arguments are applied to the triple X', U', C', as they are applied to the triple X, U, C to conclude that C is not linked with any cycle of dimension $\leqslant k$ in X' with respect to G. This means that C is a Z_k^G-set. $\square$

Lemma 2.4. *Let $C \cap U$ be a Z_k^G-set in an open U, and $C \cap V$ be a Z_k^G-set in an open V. Then $C \cap (U \cup V)$ is a Z_k^G-set in $U \cup V$.*

Proof. Let us consider the homomorphism of the Mayer–Vietoris sequences for the pairs $(U \setminus C, V \setminus C)$ and (U, V) induced by the inclusion. The homomorphisms $H_i((U \cup V) \setminus C; G) \to H_i(U \cup V; G)$ will be isomorphisms for $i \leqslant k$ by the Five-lemma. If we consider any open subset W of the union $U \cup V$, then one can consider $U' = W \cap U$, $V' = W \cap V$ and by the same arguments one obtains the isomorphism $H_i((U' \cup V') \setminus C; G) \to H_i(U' \cup V'; G)$. $\square$

Corollary 2.1. *Let ω be an open covering of an LC^k space M and C is such a compact subset of M that intersections $M \cap U$ are Z_k^G-sets in U for all $U \in \omega$. Then C is a Z_k^G-set in M.*

The following lemma is a variant of the Alexandroff Addition Theorem [1].

Lemma 2.5. *Let M be a space such that $H_{k+1}(M; G) = 0$ and F_1, F_2 be compact subspaces of M. Let z_k be a k-dimensional cycle linked with the union $F_1 \cup F_2$ in M. Then one of the following holds*:

(1) *the cycle z_k is linked either with F_1 or with F_2; or*
(2) *$F_1 \cap F_2 \neq \emptyset$ and there is a cycle z_{k+1} of dimension $k+1$ linked with $F_1 \cap F_2$ in M.*

Proof. Let $B = (M \setminus F_1) \cup (M \setminus F_2) = M \setminus (F_1 \cap F_2)$ and taking in account that $(M \setminus F_1) \cap (M \setminus F_2) = (M \setminus (F_1 \cup F_2))$, we consider the Mayer–Vietoris exact sequence for B:

$$\begin{aligned} H_{k+1}(B;G) &\to H_k\big(M \setminus (F_1 \cup F_2); G\big) \\ &\to H_k(M \setminus F_1; G) \oplus H_k(M \setminus F_2, G) \\ &\to H_k(B;G). \end{aligned}$$

If $F_1 \cap F_2$ is empty, then $B = M$, hence $H_{k+1}(B;G) = 0$. That implies the injectivity of the second homomorphism, so the image of z_k under this homomorphism is not trivial. Therefore z_k is linked with F_1 or F_2 in M.

If $F_1 \cap F_2 \neq \emptyset$, then the image of z_k may be trivial. In this situation by the exactness of the Mayer–Vietoris sequence there is z_{k+1} which covers z_k under the first homomorphism. Thus z_{k+1} is linked with the intersection $F_1 \cap F_2$ in M. □

Lemma 2.6. *Let $\{C_i\}$ be a decreasing sequence of compact sets in an LC^n-space M. If a cycle z of dimension $k \leqslant n$ is linked with each C_i in M, then the cycle z is linked with the intersection $\bigcap C_i$ in M.*

Proof. Since $M \setminus \bigcap C_i = \bigcup (M \setminus C_i)$, the lemma follows from the isomorphism of $H_k(M \setminus \bigcap C_i; G)$ and $\varinjlim H_k(M \setminus C_i; G)$. □

Definition. A cycle z is said to be *irreducibly linked* with a compact C in a space M, if z is linked with C in M and z is unlinked in M with any compact proper subset of C.

Lemma 2.7. *If a cycle z of dimension $\leqslant n$ is linked with a compact subset C in some LC^n-space, then it is irreducibly linked with some subcompactum of C.*

Proof. It follows from Zorn lemma and Lemma 2.6. □

Lemma 2.8. *If a compactum C is not linked in an LC^k and C^{k+1}-space M with any cycle of dimension $\leqslant k$ then C is a Z_k^G-set in M.*

Proof. Let U be an open subset and z an i-dimensional ($i \leqslant k$) cycle in $U \setminus C$ which is homologous to 0 in U. To prove that z is not linked with C in U. Let us consider the Mayer–Vietoris sequence for the pair $(M \setminus C, U)$.

$$H_{i+1}(M;G) \to H_i(U \setminus C; G) \to H_i(U;G) \oplus H_i(M \setminus C; G).$$

The image of z is trivial not only in $H_i(U;G)$ but also in $H_i(M \setminus C; G)$ as C does not link cycles in M. Hence the image of z is trivial in the sum $H_i(M \setminus C; G) \oplus H_i(U;G)$. Therefore z belongs to the image of $H_{i+1}(M;G)$. But this group is trivial by our hypothesis. □

Lemma 2.9. *Suppose that all points of a 0-dimensional compactum C are Z_k^G-sets in some LC^k and C^{k+1}-space M. Then C is a Z_k^G-set in M.*

Proof. By virtue of Lemma 2.8 it is sufficient to prove that C is not linked with cycles of dimension $\leqslant k$ in M. Suppose there is a cycle z of dimension $\leqslant k$ linked with C in M. By

Lemma 2.7 we can find in C a subcompact which is linked with z irreducibly. We denote this compact also by C. As C cannot be a singleton by our assumptions, it is possible to present it as the union $C_1 \cup C_2$ where C_1 and C_2 are disjoint proper subcompacta of C. Now z cannot links both of them which contradicts Lemma 2.5 on the linking with the unions. $\square$

Lemma 2.10. *If all points of an n-dimensional compactum C are Z_k^G-sets in some LC^k and C^{k+1}-space M, then C is a Z_{k-n}^G-set in M.*

Proof. The proof is by induction on n. The case $n = 0$ was considered in Lemma 2.9. Suppose that the lemma holds for all compacta of dimension $< n$. By virtue of Lemma 2.8, it suffices to prove that C is not linked with any cycle of dimension $\leqslant k - n$ in M. Let z be a cycle of dimension $i \leqslant k - n$ linked with C. Let C' be a subcompactum of C which is irreducibly linked with z. As $\dim C' \leqslant n$ one can decompose C' as the union of proper subcompacta $C_1 \cup C_2$ and the dimension of the intersection is less than n. Hence, by induction hypothesis, the intersection is a Z_{k-n+1}^G-set and thus is not linked with any cycle of dimension $i + 1$ and by virtue of Lemma 2.5 z is linked with one of the C_i. This contradicts the irreducibility of C'. $\square$

By ΣM we denote the *suspension* of the space M, which is the quotient space $M \times [-1, 1]/M \times \{1\} \cup M \times \{-1\}$. We identify M with $M \times \{0\}$.

Lemma 2.11. *For every compactum C in an LC^k-space M and for any $i \leqslant k$ there is a natural isomorphism $H_{i+1}(\Sigma M, \Sigma M \setminus C; G) \approx H_i(M, M \setminus C; G)$.*

Proof. Let us consider the following sets

$$\Sigma^+ M = \bigl\{(x,t) \in \Sigma M \mid t \geqslant 0\bigr\} \quad \text{and} \quad \Sigma^- M = \bigl\{(x,t) \in \Sigma M \mid t \leqslant 0\bigr\}.$$

Then M coincides with its intersection. As $\Sigma^{\pm} M \setminus C$ are contractible the Mayer–Vietoris sequence for this pair gives the isomorphism

$$H_{i+1}(\Sigma M \setminus C; G) \approx H_i(M \setminus C; G) \quad \text{for } i \leqslant k.$$

And since $H_{i+1}(\Sigma M; G) \approx H_i(M; G)$ for all i, by the Five-lemma the morphism of the exact sequences of the pairs $(M, M \setminus C)$ and $(\Sigma M, \Sigma M \setminus C)$ gives the isomorphism $H_{i+1}(\Sigma M, \Sigma M \setminus C; G) \approx H_i(M, M \setminus C; G)$. $\square$

Theorem 2.1. *Suppose that all points of an n-dimensional compactum C are Z_k^G-sets in some LC^{k+1}-space M. Then C is a Z_{k-n}^G-set in M.*

Proof. By Lemmas 2.11, 2.1 and the remark after that lemma and Lemma 2.2, the points of C are Z_{k+1}^G-sets in ΣM. Let us consider the contractible open set $\Sigma M \setminus (M \times \{1\})$. Then all points of C are Z_{k+1}^G-sets in $\Sigma M \setminus (M \times \{1\})$. By Lemma 2.10 C is a Z_{k+1-n}^G-set in $\Sigma M \setminus (M \times \{1\})$. Also, by Lemma 2.3 C is a Z_{k+1-n}^G-set in ΣM. Now we prove that C

is a Z^G_{k-n}-set in M. Suppose the contrary holds. Let $H_i(U, U \setminus C; G) \neq 0$ for some open $U \subset M$ and for some $i \leqslant k-n+1$. By Lemma 2.11

$$H_{i+1}(\Sigma U, \Sigma U \setminus C; G) \neq 0.$$

By the excision axiom

$$H_{i+1}(\Sigma U, \Sigma U \setminus C; G) \approx H_{i+1}\big(U \times (-1,1), \big(U \times (-1,1)\big) \setminus C; G\big).$$

However, $U \times (-1,1)$ is an open subset of ΣM. This contradicts that C is a Z^G_{k+1-n}-set in ΣM. □

The set which is a Z^G_k-set for all k is denoted by Z^G_∞-set. The above Theorem 2.1 and Lemma 2.2 give us the following result for ANR spaces, which was independently obtained by Edwards.

Corollary 2.2. *A compact subset C of an ANR space M is a Z^G_∞-set for an Abelian group G iff $H_*(M, M \setminus x; G) = 0$ for all $x \in C$.*

3. Menger manifolds

Definition. A closed subset $X \subset M$ is said to be a Z-*set* in M if for any $\varepsilon > 0$ there is a map $f : M \to M \setminus X$ ε-close to the identity map id_M.

By μ_n we denote the n-dimensional universal Menger compactum.

Lemma 3.1. *Let X be a Z-set in an LC^{n-1}-space M. Then X is a Z^G_{n-1}-set for every Abelian group G. In particular the conclusion holds for μ_n.*

Proof. Let z be a k-cycle over G in $V = U \setminus X$, which is homologous to 0 in U ($k \leqslant n-1$). We prove that z is homologically trivial in V. Since z is homologically trivial in U there exists a chain w_{k+1} in U such that $z = \partial w_{k+1}$. Since X is a Z-set there exists a map $h : U \to U$ sufficiently close to the identity such that $h(U) \cap X = \emptyset$. We get a cycle $z' = h_*(z) = h_*(\partial w_{k+1}) = \partial h_*(w_{k+1}) = \partial w'_{k+1}$ that is homologically trivial in V. Since the cycles z and z' are homotopic in V by virtue of the local k-connectedness of M for h sufficiently close to the identity, they are homologous in V. This means that there exists a chain w''_{k+1} in V such that $\partial w''_{k+1} = z - z'$. Let $w'''_{k+1} = w''_{k+1} + w'_{k+1}$. Then $\partial w'''_{k+1} = z$. Thus z is homologically trivial in V. □

Theorem 3.1. *Let M be a μ_n-manifold. Then every k-dimensional compact $C \subset M$ is a Z^G_{n-k-2}-set in M for every Abelian group G.*

Proof. Every point of a Menger manifold is known to be a Z-set [2]. Thus by Lemma 3.1 all points of Menger manifolds are also homology Z_{n-1}-sets and Theorem 2.1 can be applied. □

Lemma 3.2. *Let C' be a closed subspace of a 0-dimensional compactum C. Let M be a complete metric space without isolated points. Then every embedding of C' into M can be extended to an embedding of C into M.*

Proof. The proof is based on a standard Baire category argument applied to the space of mappings of C into M fixed on C'. $\square$

Lemma 3.3. *Countable union of compact Z_k^G-sets in a complete metric LC^k-space is a Z_k^G-set.*

Proof. In this proof we will use Vietoris homology, which coincides with singular homology up to dimension k by [4]. Let $\{F_i\}$ be a countable family of compact Z_k^G-sets. Let z_k be a k-dimensional cycle lying in the complement of $U \setminus \bigcup F_i$ for some open $U \subset X$ and assume that the cycle z_k bounds in U. Since F_1 is a Z_k^G-set one has that z_k bounds in $U \setminus F_1$. Let us fix a compact $C_1 \subset U \setminus F_1$ supporting the chain bounded by z_k. Denote by ε_1 the distance between C_1 and F_1. Consider now $O_{\varepsilon_1/2}C_1 \subset U \setminus F_1$. Then the cycle z_k bounds in $O_{\varepsilon_1/2}C_1$. Since F_2 is a Z_k^G-set, z_k bounds in $O_{\varepsilon_1/2}C_1 \setminus F_2$. There is a compact set $C_2 \subset O_{\varepsilon_1/2}C_1 \setminus F_2$ such that C_2 supports the chain bounded by z_k. Choose ε_2 smaller than the distance between C_2 and F_2 and $\varepsilon_1/2$.

By induction we construct sequences $\{C_i\}$ and $\{\varepsilon_i\}$ such that

(1) z_k bounds a chain which is supported by a compact set C_i such that $C_i \subset O_{\varepsilon_{i-1}/2}C_{i-1} \setminus F_i$;
(2) $0 < \operatorname{dist}(C_i, F_i) < \varepsilon_i$;
(3) $\varepsilon_{i+1} < \varepsilon_i/2$.

Denote by $C_\infty = \overline{\lim}\, C_i$. The completeness of our space implies that each of its neighbourhood contains all but finitely many of F_i. Hence C_∞ supports the chain bounded by z_k. Moreover $C_\infty \cap (\bigcup F_i) = \emptyset$. Hence z_k is trivial in $U \setminus \bigcup F_i$. $\square$

Theorem 3.2. *Any 0-dimensional compactum is μ-uncolinked iff it is countable.*

Proof. By Lemma 3.3 a countable compactum is a Z_{n-1}^G-set in any n-dimensional Menger manifold M. Therefore it is μ-uncolinked. If C is an uncountable compactum then it contains a Cantor set, and hence contains homeomorphic copy of each 0-dimensional compact set. But every n-dimensional Menger manifold M contains a colinked 0-dimensional compactum D such that the cycle with which D is linked can be chosen as an $(n-2)$-cycle by virtue of Theorem 1.1. Let z be the cycle of dimension $n-2$ in M linked with D. And let F be a compact support of z which does not intersects with D. Choose a subset C' in C homeomorphic to D. Fix a homeomorphism $f : C' \to D$. By Lemma 3.2 we can extend f to an embedding of X into $M \setminus D$. Thus the cycle z links the image of C in M. $\square$

Dimension $\operatorname{ind}_\sigma$. Let us introduce a dimension type invariant $\operatorname{ind}_\sigma$ which is defined for all compact spaces by induction in the following way:

(1) $\operatorname{ind}_\sigma X = -1$ iff X is finite or countable;
(2) $\operatorname{ind}_\sigma X \leqslant n$ iff $\{U \subset X \mid \operatorname{ind}_\sigma \partial U < n\}$ forms an open base of X.

The following lemma lists the properties of ind_σ which are of importance for us:

Lemma 3.4.
(1) $\mathrm{ind}_\sigma X \leqslant \dim X$ *for every compact* X;
(2) $X' \subset X$ *implies* $\mathrm{ind}_\sigma X' \leqslant \mathrm{ind}_\sigma X$;
(3) $\mathrm{ind}_\sigma P < \dim P$ *for every polyhedron* P.

Proof. The first property is an immediate consequence of the Urysohn equality $\mathrm{ind}\, X = \dim X$. The second is proved in the same way as the one for usual inductive dimension. To prove the third one let us remark that an n-dimensional polyhedron has an open base consisting of interiors of finite subpolyhedra. All elements of such base have polyhedral boundaries of dimension $< n$. Since 0-dimensional polyhedra are finite and have negative ind_σ-dimension, the third property may be proved by induction on the dimension of polyhedra. □

Theorem 3.3. *In each μ_n-manifold every compactum C with* $\mathrm{ind}_\sigma C < k$ *is a Z^G_{n-k-1}-set for every Abelian group G.*

Proof. Since every μ_n-manifold has an open covering by sets whose closures are homeomorphic to μ_n, by virtue of Corollary 2.1 it is sufficient to prove our theorem for the Menger cube μ_n. The case $k = 0$ directly follows from Theorem 3.2, so we may assume that $k \geqslant 1$. By Lemma 2.8 it is sufficient to prove that any compactum C of $\mathrm{ind}_\sigma C < n$ is not linked with any cycle of dimension $\leqslant n - k - 1$ in μ_n.

We will prove the theorem by induction on $\mathrm{ind}_\sigma C$. As was mentioned above, theorem holds for $k = 0$.

Suppose that the theorem holds for $\mathrm{ind}_\sigma C \leqslant (k-1)$ and let us prove the conclusion for $\mathrm{ind}_\sigma C = k$. Let z be a $(n-k-1)$-cycle in $V = \mu_n \setminus C$. We need to prove that z is homologically trivial in V. Assume it is not. Then z is linked with C in μ_n. By Lemma 2.7 there is a compact $C' \subset C$ which is irreducibly linked with z in μ_n. Then $\mathrm{ind}_\sigma C' \leqslant k$ and there is an open proper subset U of C', such that $\mathrm{ind}_\sigma \partial U < k$. Denote by C_1 the closure of U and by C_2 the closure of $C' \setminus U$. Then $C_1 \cup C_2 = C'$ and $C_1 \cap C_2 = \partial U$. By irreducibility of C' the cycle z does not link C_1 and C_2. By Lemma 2.5 in this case there is a $(n-k)$-dimensional cycle linked with intersection $C_1 \cap C_2 = \partial U$. But this contradicts the induction hypothesis because $\mathrm{ind}_\sigma \partial U \leqslant k - 1$. □

Theorem 3.3 implies Theorems 1.2 and 1.3 stated in the introduction. It also has the following corollary:

Corollary 3.1. *The inequality* $\mathrm{ind}_\sigma C < \dim C$ *implies that the compactum C is μ-uncolinked.*

This corollary is our main tool in establishing of the μ-uncolinkedness of compacta. Since a one-dimensional ANR-compactum is a local dendrite [5], it is separated by finite sets. The same is true for the triangular Sierpinski curve (Sierpinski gasket). This implies that their ind_σ-dimension is 0. Therefore one gets the following corollaries:

Corollary 3.2. *Every* 1*-dimensional compact ANR is* μ*-uncolinked.*

Corollary 3.3. *The Sierpinski gasket is* μ*-uncolinked.*

Every Pontrjagin surface Π can be presented as a finite union of arbitrarily small closed sets whose boundaries are homeomorphic to the circumference S^1 (see [3]). This implies the inequality $\mathrm{ind}_\sigma \Pi \leqslant 1$. Hence one obtains the following corollary:

Corollary 3.4. *All Pontrjagin surfaces are* μ*-uncolinked.*

On the other hand one has the following:

Example 3.1. The Sierpinski carpet is μ-colinked.

Proof. There is a plane which intersects the standard Menger curve by Sierpinski carpet. So Sierpinski carpet can separate μ_1, i.e., can link with a 0-dimensional cycle. Thus it is colinked. □

We conclude with the following question.

Question. Is it true that every μ-uncolinked compactum X satisfies the inequality $\mathrm{ind}_\sigma X < \dim X$?

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